

Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p -Laplacian

Giuseppina D'Agui^a, Shapour Heidarkhani^{b,c} and Giovanni Molica Bisci^{d*}

^aDepartment of Civil, Information Technology,
Construction, Environmental Engineering and Applied Mathematics,
University of Messina, C/da di Dio,
98166, Messina, Italy
e-mail: dagui@unime.it

^bDepartment of Mathematics, Faculty of Sciences, Razi University,
67149 Kermanshah, Iran

^cSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box: 19395-5746, Tehran, Iran
e-mail address: s.heidarkhani@razi.ac.ir

^dDipartimento MECMAT, University of Reggio Calabria, Via Graziella, Feo di Vito,
89124, Reggio Calabria, Italy
e-mail: gmolica@unirc.it

Abstract

The existence of three distinct weak solutions for a perturbed mixed boundary value problem involving the one-dimensional p -Laplacian operator is established under suitable assumptions on the nonlinear term. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces.

Keywords: Multiple solutions; perturbed mixed boundary value problem; critical point theory; variational methods.

AMS subject classification: 34B15.

1 Introduction

Consider the following perturbed mixed boundary value problem

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(x, u) + \mu g(x, u) & \text{in }]a, b[\\ u(a) = u'(b) = 0, \end{cases} \quad (1)$$

where $p > 1$, $\lambda > 0$ and $\mu \geq 0$ are real numbers, $a, b \in \mathbb{R}$ with $a < b$, $\rho, s \in L^\infty([a, b])$ with $\rho_0 = \operatorname{ess\,inf}_{x \in [a, b]} \rho(x) > 0$, $s_0 = \operatorname{ess\,inf}_{x \in [a, b]} s(x) \geq 0$ and $f, g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ are two L^1 -Carathéodory function.

*Corresponding author.

Using two kinds of three critical points theorems obtained in [4, 8] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three weak solutions for the problem (1); see Theorems 3.1 and 3.2. These theorems have been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [5, 6, 14, 16, 17].

Existence and multiplicity of solutions for mixed boundary value problems have been studied by several authors and, for an overview on this subject, we refer the reader to the papers [2, 3, 12, 15, 18]. We also refer the reader to the papers [7, 9, 10, 11] in which the existence of multiple solutions is ensured.

A special case of Theorem 3.1 is the following theorem.

Theorem 1.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(\eta) > 0$ for some $\eta > 0$ and $F(\xi) \geq 0$ in $[0, \eta]$ and*

$$\liminf_{\xi \rightarrow 0} \frac{F(\xi)}{\xi^p} = \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, there is $\lambda^ > 0$ such that for each $\lambda > \lambda^*$ and for every L^1 -Carathéodory function $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the asymptotical condition*

$$\limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [a, b]} \int_0^t g(x, s)ds}{t^p} < +\infty,$$

there exists $\delta_{\lambda, g}^ > 0$ such that, for each $\mu \in [0, \delta_{\lambda, g}^*[,$ the problem*

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(u) + \mu g(x, u) & \text{in }]a, b[\\ u(a) = u'(b) = 0, \end{cases}$$

admits at least three weak solutions.

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 1.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^2} = 0,$$

and

$$\int_0^1 f(\xi)d\xi < \frac{1}{222} \int_0^2 f(\xi)d\xi.$$

Then, for every $\lambda \in \left[\frac{37}{\int_0^2 f(\xi)d\xi}, \frac{1}{6 \int_0^1 f(\xi)d\xi} \right[$ and for every L^1 -Carathéodory function $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the condition

$$\limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [0, 1]} \int_0^t g(x, s)ds}{t^3} < +\infty,$$

there exists $\delta_{\lambda,g}^* > 0$ such that, for each $\mu \in [0, \delta_{\lambda,g}^*]$, the problem

$$\begin{cases} -(|u'|u')' + |u|u = \lambda f(u) + \mu g(x, u) & \text{in }]0, 1[\\ u(0) = u'(1) = 0, \end{cases}$$

admits at least three weak solutions.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple weak solutions for the eigenvalue problem (1).

2 Preliminaries

Our main tools are the following three critical points theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([8], Theorem 2.6). *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that*

$$(a_1) \quad \frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})},$$

$$(a_2) \quad \text{for each } \lambda \in \Lambda_r := \left[\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right] \text{ the functional } \Phi - \lambda\Psi \text{ is coercive.}$$

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X .

Theorem 2.2 ([4], Theorem 3.3). *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^* , $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s \in [0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\bar{v} \in X$, with $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$, such that

$$(b_1) \quad \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{r_1} < \frac{2}{3} \frac{\Psi(\bar{v})}{\Phi(\bar{v})};$$

$$(b_2) \quad \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2])} \Psi(u)}{r_2} < \frac{1}{3} \frac{\Psi(\bar{v})}{\Phi(\bar{v})}.$$

Then, for each $\lambda \in \left[\frac{3}{2} \frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \min \left\{ \frac{r_1}{\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u \in \Phi^{-1}(]-\infty, r_2])} \Psi(u)} \right\} \right]$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2])$.

In order to study the problem (1), the variational setting is the space

$$X := \left\{ u \in W^{1,p}([a, b]) : u(a) = 0 \right\}$$

endowed with the norm

$$\|u\| := \left(\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx \right)^{1/p}.$$

We observe that the norm $\|\cdot\|$ is equivalent to the usual one.

It is well known that $(X, \|\cdot\|)$ is compactly embedded in $(C^0([a, b]), \|\cdot\|_\infty)$ and

$$\|u\|_\infty \leq \frac{(b-a)^{(p-1)/p}}{\rho_0^{1/p}} \|u\|, \quad (2)$$

for every $u \in X$.

We need the following proposition in the proof of Theorem 3.1.

Proposition 2.3. *Let $T : X \rightarrow X^*$ be the operator defined by*

$$T(u)v = \int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

for every $u, v \in X$. Then T admits a continuous inverse on X^* .

Proof. In the proof, we use C_1, C_2, \dots, C_9 to denote suitable positive constants. For any $u \in X \setminus \{0\}$,

$$\begin{aligned} \lim_{\|u\| \rightarrow \infty} \frac{\langle T(u), u \rangle}{\|u\|} &= \lim_{\|u\| \rightarrow \infty} \frac{\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx}{\|u\|} \\ &= \lim_{\|u\| \rightarrow \infty} \frac{\|u\|^p}{\|u\|} \\ &= \lim_{\|u\| \rightarrow \infty} \|u\|^{p-1} = \infty. \end{aligned}$$

Thus, the map T is coercive.

Now, taking into account (2.2) in [19], we see that

$$\begin{aligned} &\langle T(u) - T(v), u - v \rangle \\ &\geq \begin{cases} C_1 \int_a^b (\rho(x)|u'(x) - v'(x)|^p + s(x)|u(x) - v(x)|^p) dx & \text{if } p \geq 2, \\ C_2 \int_a^b \left(\frac{\rho(x)|u'(x) - v'(x)|^2}{(|u'(x)| + |v'(x)|)^{2-p}} + \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} \right) dx & \text{if } 1 < p < 2. \end{cases} \end{aligned} \quad (3)$$

At this point, if $p \geq 2$, then it follows that

$$\langle T(u) - T(v), u - v \rangle \geq C_1 \|u - v\|^p,$$

so T is uniformly monotone. By [20, Theorem 26.A (d)], T^{-1} exists and is continuous on X^* .

On the other hand, if $1 < p < 2$, by Hölder's inequality, we obtain

$$\begin{aligned} \int_a^b s(x)|u(x) - v(x)|^p dx &\leq \left(\int_a^b \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} dx \right)^{p/2} \left(\int_a^b s(x)(|u(x)| + |v(x)|)^p dx \right)^{(2-p)/2} \\ &\leq C_3 \left(\int_a^b \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} dx \right)^{p/2} \left(\int_a^b s(x)(|u(x)|^p + |v(x)|^p) dx \right)^{(2-p)/2} \end{aligned}$$

$$\leq C_4 \left(\int_a^b \frac{s(x)|u(x) - v(x)|^2}{(|u(x)| + |v(x)|)^{2-p}} dx \right)^{p/2} (\|u\| + \|v\|)^{(2-p)p/2}. \quad (4)$$

Similarly, one has

$$\int_a^b \rho(x)|u'(x) - v'(x)|^p dx \leq C_5 \left(\int_a^b \frac{\rho(x)|u'(x) - v'(x)|^2}{(|u'(x)| + |v'(x)|)^{2-p}} dx \right)^{p/2} (\|u\| + \|v\|)^{(2-p)p/2}. \quad (5)$$

Then, relation (3) together with (4) and (5), yields

$$\begin{aligned} & \langle T(u) - T(v), u - v \rangle \\ & \geq \frac{C_6}{(\|u\| + \|v\|)^{2-p}} \left(\left(\int_a^b \rho(x)|u'(x) - v'(x)|^p dx \right)^{2/p} + \left(\int_a^b s(x)|u(x) - v(x)|^p dx \right)^{2/p} \right) \\ & \geq \frac{C_7}{(\|u\| + \|v\|)^{2-p}} \left(\int_a^b \rho(x)|u'(x) - v'(x)|^p dx + \int_a^b s(x)|u(x) - v(x)|^p dx \right)^{2/p} \\ & \geq C_8 \frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}}. \end{aligned}$$

Thus, T is strictly monotone. By [20, Theorem 26.A (d)], T^{-1} exists and is bounded. Moreover, given $g_1, g_2 \in X^*$, by the inequality

$$\langle T(u) - T(v), u - v \rangle \geq C_8 \frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}},$$

choosing $u = T^{-1}(g_1)$ and $v = T^{-1}(g_2)$ we have

$$\|T^{-1}(g_1) - T^{-1}(g_2)\| \leq \frac{1}{C_9} (\|T^{-1}(g_1)\| + \|T^{-1}(g_2)\|)^{2-p} \|g_1 - g_2\|_{X^*}.$$

So T^{-1} is continuous. This completes the proof. \square

We use the following notations:

$$\|\rho\|_\infty := \operatorname{ess\,sup}_{x \in [a,b]} \rho(x), \quad \|s\|_\infty := \operatorname{ess\,sup}_{x \in [a,b]} s(x).$$

Corresponding to f and g we introduce the functions $F : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ and $G : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, respectively, as follows

$$F(x, t) := \int_0^t f(x, \xi) d\xi, \quad \forall (x, t) \in [a, b] \times \mathbb{R}$$

and

$$G(x, t) := \int_0^t g(x, \xi) d\xi, \quad \forall (x, t) \in [a, b] \times \mathbb{R}.$$

Moreover, set $G^\theta := \int_{[a,b]} \max_{|t| \leq \theta} G(x, t) dt$, for every $\theta > 0$ and $G_\eta := \inf_{[a,b] \times [0, \eta]} G$, for every $\eta > 0$.

If g is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

We mean by a (weak) solution of problem (1), any function $u \in X$ such that

$$\begin{aligned} & \int_a^b \rho(x)|u'(x)|^{p-2} u'(x) v'(x) dx + \int_a^b s(x)|u(x)|^{p-2} u(x) v(x) dx \\ & - \lambda \int_a^b f(x, u(x)) v(x) dx - \mu \int_a^b g(x, u(x)) v(x) dx = 0, \end{aligned}$$

for every $v \in X$.

3 Main results

Put

$$k := \frac{2(p+1)\rho_0}{2^p(p+1)\|\rho\|_\infty + (p+2)(b-a)^p\|s\|_\infty}, \quad (6)$$

Following the construction given in [6], in order to introduce our first result, fixing two positive constants θ and η such that

$$\frac{\eta^p}{k \int_{\frac{a+b}{2}}^b F(x, \eta) dx} < \frac{\theta^p}{\int_a^b \sup_{|t| \leq \theta} F(x, t) dx},$$

and taking

$$\lambda \in \Lambda := \left[\frac{\rho_0 \eta^p}{pk(b-a)^{p-1} \int_{\frac{a+b}{2}}^b F(x, \eta) dx}, \frac{\rho_0 \theta^p}{p(b-a)^{p-1} \int_a^b \sup_{|t| \leq \theta} F(x, t) dx} \right],$$

set $\delta_{\lambda, g}$ given by

$$\min \left\{ \frac{\rho_0 \theta^p - \lambda p(b-a)^{p-1} \int_a^b \sup_{|t| \leq \theta} F(x, t) dx}{p(b-a)^{p-1} G^\theta}, \frac{\rho_0 \eta^p - \lambda pk(b-a)^{p-1} \int_{\frac{a+b}{2}}^b F(x, \eta) dx}{pk(b-a)^p G_\eta} \right\} \quad (7)$$

and

$$\bar{\delta}_{\lambda, g} := \min \left\{ \delta_{\lambda, g}, \frac{1}{\max \left\{ 0, \frac{p(b-a)^p}{\rho_0} \limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [a, b]} G(x, t)}{t^p} \right\}} \right\}, \quad (8)$$

where we read $\rho/0 = +\infty$, so that, for instance, $\bar{\delta}_{\lambda, g} = +\infty$ when

$$\limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [a, b]} G(x, t)}{t^p} \leq 0,$$

and $G_\eta = G^\theta = 0$.

Now, we formulate our main result.

Theorem 3.1. Assume that there exist two positive constants θ and η with $\theta < \eta$ such that

$$(A_1) \int_a^{\frac{a+b}{2}} F(x, \xi) dx > 0, \text{ for each } \xi \in [0, \eta];$$

$$(A_2) \frac{\int_a^b \sup_{|t| \leq \theta} F(x, t) dx}{\theta^p} < k \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx}{\eta^p};$$

$$(A_3) \limsup_{|t| \rightarrow +\infty} \frac{\sup_{x \in [a, b]} F(x, t)}{t^p} \leq 0.$$

Then, for each $\lambda \in \Lambda$ and for every L^1 -Carathéodory function $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the condition

$$\limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [a, b]} G(x, t)}{t^p} < +\infty,$$

there exists $\bar{\delta}_{\lambda,g} > 0$ given by (8) such that, for each $\mu \in [0, \bar{\delta}_{\lambda,g}[$, the problem (1) admits at least three distinct weak solutions in X .

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{p} \|u\|^p$$

and

$$\Psi(u) = \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions.

It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b [f(x, u(x)) + \frac{\mu}{\lambda} g(x, u(x))] v(x) dx,$$

for every $v \in X$ as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \rightarrow X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X . For this end, for fixed $u \in X$, let $u_n \rightarrow u$ weakly in X as $n \rightarrow \infty$, then u_n converges uniformly to u on $[a, b]$ as $n \rightarrow \infty$; see [20]. Since f, g are L^1 -Carathéodory functions, f, g are continuous in \mathbb{R} for every $x \in [a, b]$, so

$$f(x, u_n) + \frac{\mu}{\lambda} g(x, u_n) \rightarrow f(x, u) + \frac{\mu}{\lambda} g(x, u),$$

as $n \rightarrow \infty$. Hence $\Psi'(u_n) \rightarrow \Psi'(u)$ as $n \rightarrow \infty$. Thus we proved that Ψ' is strongly continuous on X , which implies that Ψ' is a compact operator by Proposition 26.2 of [20].

Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_a^b r(x) |u'(x)|^{p-2} u'(x) v'(x) dx + \int_a^b s(x) |u(x)|^{p-2} u(x) v(x) dx,$$

for every $v \in X$, while Proposition 2.3 gives that Φ' admits a continuous inverse on X^* . Furthermore, Φ is sequentially weakly lower semicontinuous. Clearly, the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$.

Put $r := \frac{\rho_0 \theta^p}{p(b-a)^{p-1}}$, and

$$w(x) := \begin{cases} \frac{2\eta}{b-a}(x-a) & \text{if } x \in [a, \frac{a+b}{2}[\\ \eta & \text{if } x \in [\frac{a+b}{2}, b]. \end{cases} \quad (9)$$

It is easy to see that $w \in X$ and, in particular, one has

$$\|w\|^p = \frac{2^p \eta^p}{(b-a)^p} \int_a^{\frac{a+b}{2}} \rho(x) dx + \frac{2^p \eta^p}{(b-a)^p} \int_a^{\frac{a+b}{2}} (x-a)^p s(x) dx + \eta^p \int_{\frac{a+b}{2}}^b s(x) dx.$$

Taking into account $0 < \theta < \eta$, using (6), we observe that

$$0 < r < \Phi(w) < \frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}.$$

Bearing in mind relation (2), we see that

$$\begin{aligned} \Phi^{-1}(]-\infty, r]) &= \{u \in X; \Phi(u) \leq r\} \\ &= \left\{ u \in X; \frac{\|u\|^p}{p} \leq r \right\} \\ &\subseteq \{u \in X; |u(x)| \leq \theta \text{ for each } x \in [a, b]\}, \end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}([-\infty, r])} \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx \\ &\leq \int_a^b \sup_{|t| \leq \theta} F(x, t) dx + \frac{\mu}{\lambda} G^\theta.\end{aligned}$$

On the other hand, by using condition (A_1) , since $0 \leq w(x) \leq \eta$ for each $x \in [a, b]$, we infer

$$\begin{aligned}\Psi(w) &\geq \int_{\frac{a+b}{2}}^b F(x, \eta) dx + \frac{\mu}{\lambda} \int_a^b G(x, w(x)) dx \\ &\geq \int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} \inf_{[a, b] \times [0, \eta]} G \\ &= \int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} &= \frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r} \\ &\leq \frac{\int_a^b \sup_{|t| \leq \theta} F(x, t) dx + \frac{\mu}{\lambda} G^\theta}{\frac{\rho_0 \theta^p}{p(b-a)^{p-1}}},\end{aligned}\tag{10}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + \frac{\mu}{\lambda} \int_a^b G(x, w(x)) dx}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}} \\ &\geq \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}}.\end{aligned}\tag{11}$$

Since $\mu < \delta_{\lambda, g}$, one has

$$\mu < \frac{\rho_0 \theta^p - \lambda p(b-a)^{p-1} \int_a^b \sup_{|t| \leq \theta} F(x, t) dx}{p(b-a)^{p-1} G^\theta},$$

this means

$$\frac{\int_a^b \sup_{|t| \leq \theta} F(x, t) dx + \frac{\mu}{\lambda} G^\theta}{\frac{\rho_0 \theta^p}{p(b-a)^{p-1}}} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\rho_0 \eta^p - \lambda pk(b-a)^{p-1} \int_{\frac{a+b}{2}}^b F(x, \eta) dx}{pk(b-a)^p G_\eta},$$

this means

$$\frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}} > \frac{1}{\lambda}.$$

Then,

$$\frac{\int_a^b \sup_{|t| \leq \theta} F(x, t) dx + \frac{\mu}{\lambda} G^\theta}{\frac{\rho_0 \theta^p}{p(b-a)^{p-1}}} < \frac{1}{\lambda} < \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}}. \quad (12)$$

Hence from (10)-(12), we observe that the condition (a_1) of Theorem 2.1 is satisfied.

Finally, since $\mu < \bar{\delta}_{\lambda, g}$, we can fix $l > 0$ such that

$$\limsup_{|t| \rightarrow \infty} \frac{\sup_{x \in [a, b]} G(x, t)}{t^p} < l,$$

and $\mu l < \frac{\rho_0}{p(b-a)^p}$.

Therefore, there exists a function $h \in L^1([a, b])$ such that

$$G(x, t) \leq lt^p + h(x), \quad (13)$$

for every $x \in [a, b]$ and $t \in \mathbb{R}$.

Now, fix $0 < \epsilon < \frac{\rho_0}{p(b-a)^p \lambda} - \frac{\mu l}{\lambda}$. From (A_3) there is a function $h_\epsilon \in L^1([a, b])$ such that

$$F(x, t) \leq \epsilon t^p + h_\epsilon(x), \quad (14)$$

for every $x \in [a, b]$ and $t \in \mathbb{R}$.

Taking (2) into account, it follows that, for each $u \in X$,

$$\begin{aligned} \Phi(u) - \lambda \Psi(u) &= \frac{1}{p} \|u\|^p - \lambda \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx \\ &\geq \frac{1}{p} \|u\|^p - \lambda \epsilon \int_a^b u^p(x) dx - \lambda \|h_\epsilon\|_1 - \mu l \int_a^b u^p(x) dx - \mu \|h\|_1 \\ &\geq \left(\frac{1}{p} - \lambda \frac{(b-a)^p}{\rho_0} \epsilon - \mu \frac{(b-a)^p}{\rho_0} l \right) \|u\|^p - \lambda \|h_\epsilon\|_1 - \mu \|h\|_1, \end{aligned}$$

and thus

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - \lambda \Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda \Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is verified.

By using relations (10) and (12) one also has

$$\lambda \in \left[\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right].$$

Finally, Theorem 2.1 (with $\bar{x} = w$) ensures the conclusion. \square

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested. In such a case f and g are supposed to be nonnegative.

For our goal, let us fix positive constants θ_1, θ_2 and η such that

$$\frac{3}{2} \frac{\eta^p}{k \int_{\frac{a+b}{2}}^b F(x, \eta) dx} < \min \left\{ \frac{\theta_1^p}{\int_a^b \sup_{|t| \leq \theta_1} F(x, t) dx}, \frac{\theta_2^p}{2 \int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx} \right\},$$

and taking

$$\lambda \in \Lambda := \left[\frac{3}{2} \frac{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}}{\int_{\frac{a+b}{2}}^b F(x, \eta) dx}, \frac{\rho_0}{p(b-a)^{p-1}} \min \left\{ \frac{\theta_1^p}{\int_a^b \sup_{|t| \leq \theta_1} F(x, t) dx}, \frac{\theta_2^p}{2 \int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx} \right\} \right].$$

With the above notations we have the following multiplicity result.

Theorem 3.2. *Let $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition $f(x, t) \geq 0$ for every $(x, t) \in [a, b] \times (\mathbb{R}^+ \cup \{0\})$. Assume that there exist three positive constants θ_1, θ_2 and η with $2^{1/p}\theta_1 < \eta < \frac{\theta_2}{2^{1/p}}$ such that assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that*

$$(B_1) \quad \frac{\int_a^b \sup_{|t| \leq \theta_1} F(x, t) dx}{\theta_1^p} < \frac{2}{3} k \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx}{\eta^p};$$

$$(B_2) \quad \frac{\int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx}{\theta_2^p} < \frac{1}{3} k \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx}{\eta^p}.$$

Then, for each $\lambda \in \Lambda$ and for every nonnegative L^1 -Carathéodory function $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, there exists $\delta_{\lambda, g}^* > 0$ given by

$$\min \left\{ \frac{\rho_0 \theta_1^p - \lambda p(b-a)^{p-1} \int_a^b \sup_{|t| \leq \theta_1} F(x, t) dx}{p(b-a)^{p-1} G^{\theta_1}}, \frac{\rho_0 \theta_2^p - \lambda p(b-a)^{p-1} \int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx}{p(b-a)^{p-1} G^{\theta_2}} \right\}.$$

such that, for each $\mu \in [0, \delta_{\lambda, g}^*]$, the problem (1) admits at least three distinct weak solutions u_i for $i = 1, 2, 3$, such that

$$0 \leq u_i(x) < \theta_2, \quad \forall x \in [a, b], \quad (i = 1, 2, 3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2) .

To this end, put w as given in (9), as well as

$$r_1 := \frac{\rho_0 \theta_1^p}{p(b-a)^{p-1}},$$

and

$$r_2 := \frac{\rho_0 \theta_2^p}{p(b-a)^{p-1}}.$$

By using condition $2^{1/p}\theta_1 < \eta < \frac{\theta_2}{2^{1/p}}$, and bearing in mind (6), we get $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since $\mu < \delta_{\lambda,g}^*$ and $G_\eta = 0$, one has

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u)}{r_1} &= \frac{\sup_{u \in \Phi^{-1}([-\infty, r_1])} \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r_1} \\ &\leq \frac{\int_a^b \sup_{|t| \leq \theta_1} F(x, t) dx + \frac{\mu}{\lambda} G^{\theta_1}}{\frac{\rho_0 \theta_1^p}{p(b-a)^{p-1}}} \\ &< \frac{1}{\lambda} < \frac{2}{3} \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}} \\ &\leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)}, \end{aligned}$$

and

$$\begin{aligned} \frac{2 \sup_{u \in \Phi^{-1}([-\infty, r_2])} \Psi(u)}{r_2} &= \frac{2 \sup_{u \in \Phi^{-1}([-\infty, r_2])} \int_a^b [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r_2} \\ &\leq \frac{2 \int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx + 2 \frac{\mu}{\lambda} G^{\theta_2}}{\frac{\rho_0 \theta_2^p}{p(b-a)^{p-1}}} \\ &< \frac{1}{\lambda} < \frac{2}{3} \frac{\int_{\frac{a+b}{2}}^b F(x, \eta) dx + (b-a) \frac{\mu}{\lambda} G_\eta}{\frac{\rho_0 \eta^p}{pk(b-a)^{p-1}}} \\ &\leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)}. \end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified.

Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption 2. of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1). We want to prove that they are nonnegative.

Let u_0 be a weak solution of problem (1). Arguing by a contradiction, assume that the set $A = \{x \in [a, b] : u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in [a, b]$. Clearly, $\bar{v} \in X$ and one has

$$\begin{aligned} &\int_a^b \rho(x) |u'_0(x)|^{p-2} u'_0(x) \bar{v}'(x) dx + \int_a^b s(x) |u_0(x)|^{p-2} u_0(x) \bar{v}(x) dx \\ &- \lambda \int_a^b f(x, u_0(x)) \bar{v}(x) dx - \mu \int_a^b g(x, u_0(x)) \bar{v}(x) dx = 0, \end{aligned}$$

for every $v \in X$.

Thus, from our sign assumptions on the data, we have

$$0 \leq \int_A \rho(x) |u'_0(x)|^p dx + \int_A s(x) |u_0(x)|^p dx \leq 0.$$

Hence, $u_0 = 0$ in A and this is absurd. Then, we deduce $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for every $x \in [a, b]$. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0, 1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0, 1]$.

By using Theorem 2.2, for every

$$\lambda \in \left[\frac{3}{2} \frac{\Phi(w)}{\Psi(w)}, \min \left\{ \frac{r_1}{\sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u)}, \frac{r_2/2}{\sup_{u \in \Phi^{-1}([-\infty, r_2])} \Psi(u)} \right\} \right],$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the weak solutions of the problem (1) and the desired conclusion is achieved. \square

Now we prove Theorems 1.1 and 1.2 in Introduction.

Proof of Theorem 1.1: Fix $\lambda > \lambda^* := \frac{2\rho_0\eta^p}{pk(b-a)^p F(\eta)}$ for some $\eta > 0$.

Recalling that

$$\liminf_{\xi \rightarrow 0} \frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset]0, +\infty[$ such that $\lim_{n \rightarrow \infty} \theta_n = 0$ and

$$\lim_{n \rightarrow \infty} \frac{\sup_{|\xi| \leq \theta_n} F(\xi)}{\theta_n^p} = 0.$$

Indeed, one has

$$\lim_{n \rightarrow \infty} \frac{\sup_{|\xi| \leq \theta_n} F(\xi)}{\theta_n^p} = \lim_{n \rightarrow \infty} \frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^p} \frac{\xi_{\theta_n}^p}{\theta_n^p} = 0,$$

where $F(\xi_{\theta_n}) = \sup_{|\xi| \leq \theta_n} F(\xi)$.

Hence, there exists $\bar{\theta} > 0$ such that

$$\frac{\sup_{|\xi| \leq \bar{\theta}} F(\xi)}{\bar{\theta}^p} < \min \left\{ \frac{kF(\eta)}{2\eta^p}; \frac{\rho_0}{p\lambda(b-a)^p} \right\}$$

and $\bar{\theta} < \eta$.

The conclusion follows by using Theorem 3.1. \square

Proof of Theorem 1.2: Our aim is to employ Theorem 3.2 by choosing $a = 0$, $b = 1$, $\rho(x) = s(x) = 1$ (for every $x \in [a, b]$) $\theta_2 = 1$ and $\eta = 2$.

Therefore, since $k = 8/37$, we see that

$$\frac{\frac{\rho_0\eta^p}{2} \frac{pk(b-a)^{p-1}}{\int_{\frac{a+b}{2}}^b F(x, \eta) dx}}{\int_0^2 f(\xi) d\xi} = \frac{37}{\int_0^2 f(\xi) d\xi}$$

and

$$\frac{\rho_0}{p(b-a)^{p-1}} \frac{\theta_2^p}{2 \int_a^b \sup_{|t| \leq \theta_2} F(x, t) dx} = \frac{1}{6 \int_0^1 f(\xi) d\xi}.$$

Moreover, since $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^2} = 0$, one has

$$\lim_{t \rightarrow 0^+} \frac{\int_0^t f(\xi) d\xi}{t^3} = 0.$$

Then, there exists a positive constant $\theta_1 < \sqrt[3]{4}$ such that

$$\frac{\int_0^{\theta_1} f(\xi) d\xi}{\theta_1^3} < \frac{\int_0^2 f(\xi) d\xi}{111},$$

and

$$\frac{\theta_1^3}{\int_0^{\theta_1} f(\xi) d\xi} > \frac{1}{2 \int_0^1 f(\xi) d\xi}.$$

Finally, a simple computation shows that all assumptions of Theorem 3.2 are fulfilled. The desired conclusion follows. \square

Acknowledgements

The authors wish to thank Professor G. Bonanno for his interesting suggestions and remarks on this subject. Research of S. Heidarkhani was in part supported by a grant from IPM(No. 91470046). G. Molica Bisci is supported by the GNAMPA Project 2012: *Esistenza e molteplicità di soluzioni per problemi differenziali non lineari*.

References

- [1] D. Averna and G. Bonanno, *A mountain pass theorem for a suitable class of functions*, Rocky Mountain J. Math. **39** (2009), 707-727.
- [2] D. Averna, S.M. Buccellato and E. Tornatore, *On a mixed boundary value problem involving the p -Laplacian*, Le Matematiche Vol. LXVI (2011)-Fasc. I, 93-104.
- [3] D. Averna and R. Salvati, *Three solutions for a mixed boundary value problem involving the one-dimensional p -laplacian*, J. Math. Anal. Appl. **298** (2004), 245-260.
- [4] G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), 3031-3059.
- [5] G. Bonanno and A. Chinnì, *Existence of three solutions for a perturbed two-point boundary value problem*, Appl. Math. Lett. **23** (2010), 807-811.
- [6] G. Bonanno and G. D'Agù, *Multiplicity results for a perturbed elliptic Neumann problem*, Abstract and Applied Analysis 2010 (2010), doi:10.1155/2010/564363, 10 pages.
- [7] G. Bonanno and G. D'Agù, *A Neumann boundary value problem for the Sturm-Liouville equation*, Appl. Math. Comput. **208** (2009), 318-327.
- [8] G. Bonanno and S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. **89** (2010), 1-10.

- [9] G. Bonanno and G. Molica Bisci, *Three weak solutions for elliptic Dirichlet problems*, J. Math. Anal. Appl. **382** (2011), 1-8.
- [10] G. Bonanno, G. Molica Bisci and V. Rădulescu, *Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces*, Nonlinear Anal. **74** (14) (2011), 4785-4795.
- [11] G. Bonanno, G. Molica Bisci and V. Rădulescu, *Multiple solutions of generalized Yamabe equations on Riemannian manifolds and applications to Emden-Fowler problems*, Nonlinear Anal. Real World Appl. **12** (2011), 2656-2665.
- [12] G. Bonanno and E. Tornatore, *Infinitely many solutions for a mixed boundary value problem*, Ann. Polon. Math. **99** (2010), 285-293.
- [13] H. Brézis, *Analyse Fonctionnelle-Théorie et Applications*, Masson, Paris, 1983.
- [14] P. Candito, G. D'Aguì, *Three solutions to a perturbed nonlinear discrete Dirichlet problem*, J. Math. Anal. Appl. **375** (2011), 594-601.
- [15] G. D'Aguì, *Existence results for a mixed boundary value problem with Sturm-Liouville equation*, Adv. Pure Appl. Math. **2** (2011), 237-248.
- [16] G. D'Aguì and A. Sciammetta, *Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions*, Nonlinear Anal. **75** (2012), 5612-5619.
- [17] S. Heidarkhani and J. Henderson, *Multiple solutions for a nonlocal perturbed elliptic problem of p -Kirchhoff type*, Communications on Applied Nonlinear Analysis **19** (3) (2012), 25-39.
- [18] R. Salvati, *Multiple solutions for a mixed boundary value problem*, Math. Sci. Res. J. **7** (2003), 275-283.
- [19] J. Simon, *Régularité de la solution d'une equation non lineaire dans \mathbb{R}^N* , in: Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977), (P. Bénilan, J. Robert, eds.), Lecture Notes in Math., 665, pp. 205-227, Springer, Berlin-Heidelberg-New York, 1978.
- [20] E. Zeidler, *Nonlinear functional analysis and its applications*, Vol. II. Berlin-Heidelberg-New York 1985.

(Received November 1, 2012)